# 3

# Green Function

**Structure**



**3.1. Introduction.** This chapter contains methods to obtain Green function for a given non-homogeneous linear second order boundary value problem and reduction of boundary value problem to Fredholm integral equation with Green function as kernel.

**31.1. Objective.** The objective of these contents is to provide some important results to the reader like:

(i) Construction of Green function.

(ii) Reduction of boundary value problem to Fredholm integral equation with Green function as kernel.

**3.1.2. Keywords.** Green function, Integral Equations, Boundary Conditions.

**3.2. Construction of Green function.** Consider a differential equation of order n

$$L(u) = p_0(x)\, u^n + p_1(x)\, u^{n-1} + p_2(x)\, u^{n-2} + ....... + p_n(x)\, u = 0 \qquad (1)$$

where the functions $p_0(x), p_1(x)$ , $p_2(x),......., p_n(x)$ are continuous on [a, b], $p_0(x) \neq 0$ on [a, b], and the boundary conditions

$$V_k(u) = \alpha_k u(a) + \alpha_k^1 u'(a) + \alpha_k^2 u''(a) + .......... + \alpha_k^{n-1} u^{n-1}(a)$$

$$+\beta_k u(b) + \beta_k^1 u'(b) + \beta_k^2 u''(b) + ........... + \beta_k^{n-1} u^{n-1}(b) \quad (2)$$

for k = 1, 2,…,n, where the linear forms $V_1, V_2,…, V_n$ in u(a), $u'$(a),…, $u^{n-1}$(a), u(b), $u'$(b), …, $u^{n-1}$(b) are linearly independent.

The homogeneous boundary value problem (1), (2) contains only a trivial solution $u(x) \equiv 0$.

Green's function of the boundary value problem (1), (2) is the function G (x, $\xi$) constructed for any point $\xi$, a < $\xi$ < b satisfying the following properties :

1. G(x, $\xi$) is continuous in x for fixed $\xi$ and has continuous derivatives with regard to x upto order $(n-2)$ inclusive for $a \le x \le b$.
2. Its $(n-1)$th derivative with regard to x at the point x = $\xi$ has a discontinuity of first kind, the jump being equal to $-\dfrac{1}{[p_0(x)]_{x=\xi}}$, that is,

$$\left\{\frac{\partial^{n-1}}{\partial x^{n-1}} G(x,\xi)\right\}_{x=\xi+0} - \left\{\frac{\partial^{n-1}}{\partial x^{n-1}} G(x,\xi)\right\}_{x=\xi-0} = -\frac{1}{p_0(\xi)} \quad (3)$$

   where $G|_{x=\xi+0}$ defines the limit of G(x, $\xi$) as $x \to \xi$ from the right and $G|_{x=\xi-0}$ defines the limit of G(x, $\xi$) as $x \to \xi$ from the left.

3. In each of the intervals [a, $\xi$) and ($\xi$, b] the function G(x, $\xi$), considered as a function of x, is a solution of the equation (1)

$$L(G) = 0 \quad (4)$$

4. The function G(x, $\xi$) satisfies the boundary conditions (2)

$$V_k(G) = 0, k = 1, 2, 3,…,n, \quad (5)$$

If the boundary value problem (1), (2) contains only the trivial solution $u(x) \equiv 0$ then the operator L contains one and only one Green's function G(x, $\xi$).

Consider $u_1(x), u_2(x),…, u_n(x)$ be linearly independent solutions of the equation L(u) = 0. From the condition 1, the unknown Green's function G(x, $\xi$) must have the representation on the intervals [a, $\xi$) and ($\xi$, b]

$$G(x, \xi) = a_1u_1(x) + a_2u_2(x) + … + a_nu_n(x), a \le x < \xi$$

and

$$G(x, \xi) = b_1u_1(x) + b_2u_2(x) + … + b_nu_n(x), \xi \le x < b,$$

where $a_1, a_2,…,a_n, b_1, b_2,…, b_n$ are some functions of $\xi$.

From the condition 1, the continuity of the function G(x, $\xi$) and of its first (n−2) derivatives with regard to x at the point x = $\xi$ yields

$$[b_1u_1(\xi) + b_2u_2(\xi) + \dots + b_nu_n(\xi)] - [a_1u_1(\xi) + a_2u_2(\xi) + \dots + a_nu_n(\xi)] = 0$$

$$[b_1u_1'(\xi) + b_2u_2'(\xi) + \dots + b_nu_n'(\xi)] - [a_1u_1'(\xi) + a_2u_2'(\xi) + \dots + a_nu_n'(\xi)] = 0$$

$$[b_1u_1''(\xi) + b_2u_2''(\xi) + \dots + b_nu_n''(\xi)] - [a_1u_1''(\xi) + a_2u_2''(\xi) + \dots + a_nu_n''(\xi)] = 0$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$[b_1u_1^{n-2}(\xi) + b_2u_2^{n-2}(\xi) + \dots + b_nu_n^{n-2}(\xi)] - [a_1u_1^{n-2}(\xi) + a_2u_2^{n-2}(\xi) + \dots + a_nu_n^{n-2}(\xi)] = 0$$

Also, $$[b_1u_1^{n-1}(\xi) + b_2u_2^{n-1}(\xi) + \dots + b_nu_n^{n-1}(\xi)] - [a_1u_1^{n-1}(\xi) + a_2u_2^{n-1}(\xi) + \dots + a_nu_n^{n-1}(\xi)] = -\frac{1}{p_0(\xi)}$$

Assume $C_k(\xi) = b_k(\xi) - a_k(\xi)$, k = 1, 2, … , n ; then the system of linear equations in $C_k(\xi)$ are obtained

$$C_1u_1(\xi) + C_2u_2(\xi) + \dots + C_nu_n(\xi) = 0$$

$$C_1u_1'(\xi) + C_2u_2'(\xi) + \dots + C_nu_n'(\xi) = 0$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$C_1u_1^{n-2}(\xi) + C_2u_2^{n-2}(\xi) + \dots + C_nu_n^{n-2}(\xi) = 0$$

$$C_1u_1^{n-1}(\xi) + C_2u_2^{n-1}(\xi) + \dots + C_nu_n^{n-1}(\xi) = -\frac{1}{p_0(\xi)} \qquad (6)$$

The determinant of the system is equal to the value of the Wronskian $W(u_1, u_2, \dots, u_n)$ at the point x = $\xi$ and is therefore different from zero.

From the boundary conditions (2), we have

$$V_k(u) = A_k(u) + B_k(u) \qquad (7)$$

where $A_k(u) = \alpha_k u(a) + \alpha^1{}_k u'(a) + \alpha^2{}_k u''(a) + \dots\dots + \alpha_k^{n-1}u^{n-1}(a)$

$$B_k(u) = \beta_k u(b) + \beta^1{}_k u'(b) + \beta^2{}_k u''(b) + \dots\dots + \beta_k^{n-1}u^{n-1}(b)$$

Using the condition 4, we have

$V_k(G) = a_1A_k(u_1) + a_2A_k(u_2) + \dots + a_nA_k(u_n) + \dots + b_1B_k(u_1) + b_2B_k(u_2) + \dots + b_nB_k(u_n) = 0$,

where k = 1, 2, …, n.

Since $a_k = b_k - c_k$, so we have

$$(b_1 - c_1)A_k(u_1) + (b_2 - c_2)A_k(u_2) + \dots + (b_n - c_n)A_k(u_n) + b_1B_k(u_1) + b_2B_k(u_2) + \dots + b_nB_k(u_n) = 0$$

$$\Rightarrow \quad b_1V_k(u_1) + b_2V_k(u_2) + \dots + b_nV_k(u_n) = c_1A_k(u_1) + c_2A_k(u_2) + \dots + c_nA_k(u_n) \qquad (8)$$

which is a linear system in the quantities $b_1, b_2, \ldots, b_n$. The determinant of the system is different from zero, that is,

$$\begin{vmatrix} V_1(u_1) & V_1(u_2) & \cdots & V_1(u_n) \\ V_2(u_1) & V_2(u_2) & \cdots & V_2(u_n) \\ \cdots & \cdots & \cdots & \cdots \\ V_n(u_1) & V_n(u_2) & \cdots & V_n(u_n) \end{vmatrix} \neq 0$$

The system of equations (8) contain a unique solution in $b_1(\xi), b_2(\xi), \ldots, b_n(\xi)$ and since

$a_k(\xi) = b_k(\xi) - c_k(\xi)$, it follows that the quantities $a_k(\xi)$ are defined uniquely.

I. If the boundary value problem (1), (2) is self – adjoint, then Green's function is symmetric, that is, $G(x, \xi) = G(\xi, x)$. The converse is true as well.

II. If at one of the extremities of an interval [a, b], the coefficient of the derivative vanishes. For example, $p_0(a) = 0$, then the natural boundary condition for the boundedness of the solution x = a is imposed, and at the other extremity the ordinary boundary condition is specified.

**3.2.1. Particular case.** We shall construct the Green's Function $G(x, \xi)$ for a given number ξ, for the second differential equation

$$L(u) + \phi(x) = 0 \tag{1}$$

where

$$L \equiv \frac{d}{dx}\left(p\,\frac{d}{dx}\right) + q \tag{2}$$

Together with the homogenous boundary conditions of the form

$$\alpha u + \beta \frac{du}{dx} = 0 \tag{3}$$

The Green's function $G(x, \xi)$ constructed for any point $\xi, a < \xi < b$ contains the following properties:

**1.** $G_1(\xi) = G_2(\xi)$; it follows that the function $G(x, \xi)$ is continuous in x for fixed ξ, in particular, continuous at the point x = ξ.

**2.** The derivatives of G(which are of finite magnitude) are continuous at every point within the range of x except at x = ξ where it is continuous so that

$$G_2'(\xi) - G_1'(\xi) - \frac{1}{p(\xi)}$$

**3.** The functions $G_1$ and $G_2$ satisfy homogenous conditions at the end points x = a and x = b respectively.

**4.** The function $G_1$ and $G_2$ satisfy the homogenous equations LG = 0 in their defined intervals except at z = ξ, that is, $LG_1 = 0, x < \xi$, $L\,G_2 = 0,\ x > \xi$.

Consider the Green's function $G(x, \xi)$ exists, then the solution of the given differential equation can be transformed to the relation

$$u(x) = \int_a^b G(x, \xi)\, \phi(\xi)\, d\xi \tag{4}$$

Consider two linearly independent solutions of the homogeneous equation $L(u) = 0$. *Let* $u = v_1(x)$ and $u = u_2(x)$ be the non-trivial solution of the equation, which satisfy the homogenous conditions at x = a and x = b respectively.

Consider the Green's functions for the problem from the conditions III and IV, in the form

$$G(x, \xi) = \begin{cases} C_1 u_1(x), & x < \xi \\ C_2\, u_2(x), & x < \xi \end{cases} \tag{5}$$

where the constant $C_1$ and $C_2$ are chosen in a manner that the conditions I and II are fulfilled. Thus, we have

$$C_2 u_2(\xi) - C_1 u_1(\xi) = 0$$

$$C_2 u_2'(\xi) - C_1 u_1'(\xi) - \frac{1}{p(\xi)} \tag{6}$$

The determinant of the system (6) is the Wronskian $W\left[u_1(\xi),\ u_2(\xi)\right]$ evaluated at the point x = ξ for linearly independent solution $u_1(x)$ and $u_2(x)$, and, hence it is different from zero $W(\xi) \neq 0$

$$W\left[u_1(\xi), u_2(\xi)\right] = \begin{vmatrix} u_1(\xi) & u_2(\xi) \\ u_1'(\xi) & u_2'(\xi) \end{vmatrix} = u_1(\xi) u_2'(\xi) - u_2(\xi) u_1'(\xi) \tag{7}$$

By using Abel's formula, we notice that the expression has the value {C/p(ξ)}, where C is a constant independent of ξ, that is,

$$u_1(\xi) u_2'(\xi) - u_2(\xi) u_1'(\xi) = \frac{C}{p(\xi)} \tag{8}$$

From the system (6), we have

$$C_1 = -\frac{1}{C}\, u_2(\xi)\ ,\ C_2 = -\frac{1}{C}\, u_1(\xi)$$

Thus the relation (5) reduces to

$$G(x, \xi) = \begin{cases} -\dfrac{1}{C} u_1(x)\, u_2(\xi), & x < \xi \\[2ex] -\dfrac{1}{C} u_1(\xi) u_2(x), & x > \xi \end{cases} \tag{9}$$

This result breaks down iff C vanishes, so that $u_1$ and $u_2$ are linearly dependent, and hence are each multiples of a certain non-trivia function U(x). In this case, the function u(x) satisfies the equation L(u) = 0 together with the end conditions at x = a, x = b.

**Converse.** The integral equation

$$u(x) = \int_a^b G(x, \xi)\, \phi(\xi)\, d\xi \qquad (10)$$

where G(x, $\xi$ ) are defined by the relation (9), satisfy the differential equation

$$L(u) + \phi(x) = 0 \qquad (11)$$

together with the prescribed boundary condition.

We know that

$$u(x) = -\frac{1}{C}\left[\int_a^x u_1(\xi)\, u_2(x)\, \phi(\xi)\, d\xi + \int_x^b u_1(x)\, u_2(\xi)\, \phi(\xi)\, d\xi\right] \qquad (12)$$

$$u'(x) = -\frac{1}{C}\left[\int_a^x u_2'(x)\, u_1(\xi)\, \phi(\xi)\, d\xi + \int_x^b u_1'(x)\, u_2(\xi)\, \phi(\xi)\, d\xi\right] \qquad (13)$$

$$u''(x) = -\frac{1}{C}\left[\int_a^x u_2''(x)\, u_1(\xi)\, \phi(\xi)\, d\xi + \int_x^b u_1''(x)\, u_2(\xi)\, \phi(\xi)\, d\xi\right]$$

$$-\frac{1}{C}\left[u_2'(x)\, u_1(x) - u_1'(x)\, u_2(x)\right]\phi(x) \qquad (14)$$

Since $L(u) \equiv p(x)u''(x) + p(x)u'(x) + q(x)\, u(x)$

Thus,

$$Lu(x) = -\frac{1}{C}\left[\int_a^x \{Lu_2(x)\}\, u_1(\xi)\, \phi(\xi)\, d\xi + \int_x^b \{Lu_2(x)\}\, u_2(\xi)\, \phi(\xi)\, d\xi\right] - \frac{1}{C}\left[p(x).\frac{C}{p(x)}\phi(x)\right]$$

Again, $u_1(x)$ and $u_2(x)$ satisfy L (u) = 0, hence the first two terms vanish identically.

So, $L\,u(x) = -\phi(x) \quad \Rightarrow \quad L\,u(x) + \phi(x) = 0$

Therefore, a function u(x) satisfying (10) also satisfies the differential equation (11)

Again from (12) and (13), we have

$$u(a) = -\frac{u_1(a)}{C}\int_a^b u_2(\xi)\, \phi(\xi)\, d\xi$$

$$u'(a) = -\frac{u_1'(b)}{C}\int_a^b u_2(\xi)\, \phi(\xi)\, d\xi$$

which shows that the function u defined by (11) satisfies the same homogeneous condition at x = a as the function $u_1$.

**Note.** Let $\phi(x) = \lambda r(x) u(x) - f(x)$.

From the differential equation (1), we have

$$Lu(x) + \lambda r(x) u(x) = f(x) \qquad (15)$$

The corresponding Fredholm integral equation becomes

$$u(x) = \lambda \int_a^b G(x, \xi)\, r(\xi)\, u(\xi)\, d\xi - \int_a^b G(x, \xi)\, f(\xi)\, d\xi \qquad (16)$$

where $G(x, \xi)$ is the Green's function.

From (9), it follows that $G(x, \xi)$ is symmetric but the kernel $K(x, \xi)$ $\{= G(x, \xi) r(\xi)\}$ is not symmetric unless r(x) is a constant.

Consider $\sqrt{\{r(x)\}}u(x) = V(x)$ with the assumption that r(x) is non – negative over (a, b). This equation (16) may be expressed as

$$\frac{V(x)}{\sqrt{r(x)}} = \lambda \int_a^b G(x, \xi) \sqrt{r(\xi)}\, V(\xi)\, d\xi - \int_a^b G(x, \xi)\, f(\xi)\, d\xi$$

or

$$V(x) = \lambda \int_a^b K(x, \xi)\, V(\xi)\, d\xi - \int_a^b K(x, \xi) \frac{f(\xi)}{\sqrt{r(\xi)}} d\xi, \qquad (17)$$

where $K(x, \xi) = \sqrt{\{r(x) r(\xi)\}}\, G(x, \xi)$ and hence possesses the same symmetry as $G(x, \xi)$.

**3.2.2. Example.** Construct an integral equation corresponding to the boundary value problem.

$$x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + (\lambda x^2 - 1)\, u = 0, \qquad (1)$$

$$u(0) = 0,\ u(1) = 0 \qquad (2)$$

**Solution.** The differential equation (1) may be written as

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) + \left(-\frac{1}{x} + \lambda x\right) u = 0.$$

or

$$\left[\frac{d}{dx}\left(x\frac{du}{dx}\right) - \frac{u}{x}\right] + \lambda x u = 0.$$

Comparing with the equation (15), we have

$$p = x,\ q = -\frac{1}{x},\ r = x \qquad (3)$$

The general solution of the homogeneous equation

$$L(u) = 0 \quad \Rightarrow \quad \left\{\frac{d}{dx}\left(x\frac{du}{dx}\right) - \frac{u}{x}\right\} = 0 \text{ is given by}$$

$$u(x) = C_1x + C_2\left(\frac{1}{x}\right)$$

Consider $u = u_1(x)$ and $u = u_2(x)$ be the non – trivial solutions of the equation, which satisfy the conditions at x = 0 and x = 1 respectively then

$$u_1(x) = x \qquad \text{and } u_2(x) = \frac{1}{x} - x .$$

The Wronskian of $u_1(x)$ and $u_2(x)$ is given by

$$W[u_1(x), u_2(x)] = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = x\left(-\frac{1}{x^2} - 1\right) - \left(\frac{1}{x} - x\right) = -\frac{2}{x}$$

So, $$u_1(x)u_2'(x) - u_2(x)u_1'(x) = -\frac{2}{x} \Rightarrow C = -2$$

Thus from the relation (19), we have

$$G(x, \xi) = \begin{cases} \frac{1}{2}\frac{x}{\xi}(1-\xi^2), & x < \xi, \\ \frac{1}{2}\frac{\xi}{x}(1-x^2), & x > \xi, \end{cases} \tag{4}$$

Therefore, from (16), the corresponding Fredholm integral equation becomes

$u(x) = \lambda\int_0^1 G(x, \xi)\, \xi\, u(\xi)\, d\xi$, where the Green's function $G(x, \xi)$ is defined by the relation (4).

**3.2.3. Example.** Construct Green's function for the homogeneous boundary value problem

$$\frac{d^4u}{dx^4} = 0 \text{ with the conditions } u(0) = u'(0) = 0,\ u(1) = u'(1) = 0.$$

**Solution.** The differential equation is given by

$$\frac{d^4u}{dx^4} = 0 \tag{1}$$

We notice that the boundary value problem contains only a trivial solution. The fundamental system of solutions for the differential equation (1) is

$$u_1(x) = 1,\ u_2(x) = x,\ u_3(x) = x^2,\ u_4(x) = x^3 \tag{2}$$

Its general solution is of the form

$$u(x) = A + Bx + Cx^2 + Dx^3,$$

where A, B, C, D are arbitrary constants. The boundary conditions give the relations for determining the constants A, B, C, D :

$$u(0) = 0 \quad \Rightarrow \quad A = 0,\ u'(0) = 0 \quad \Rightarrow \quad B = 0$$

u(1) = 0 $\Rightarrow$ A + B + C + D = 0, $u'(1) = 0$ $\Rightarrow$ B + 2C + 3D = 0

$\Rightarrow$ A = B = C = D = 0.

Thus the boundary value problem has only a zero solution $u(x) \equiv 0$ and hence we can construct a unique Green's function for it.

<u>*Construction of Green's Function:*</u> **C**onsider the unknown Green's function $G(x, \xi)$ must have the representation on the interval $[0, \xi)$ and $(\xi, 1]$.

$$G(x, \xi) = \begin{cases} a_1.1 + a_2.x + a_3.x^2 + a_4.x^3 \quad , \quad 0 \le x \le \xi \\ b_1.1 + b_2.x + b_3.x^2 + b_4.x^3 \quad , \quad \xi \le x \le 1 \end{cases} \tag{3}$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are the unknown functions of $\xi$.

Consider $C_k = b_k(\xi) - a_k(\xi)$, k = 1, 2, 3, 4,… (4)

The system of linear equations for determining the functions $C_k(\xi)$ become

$$C_1 + C_2\xi + C_3\xi^2 + C_4\xi^3 = 0$$

$$C_2 + 2C_3\xi + 3C_4\xi^2 = 0$$

$$2C_3 + 6C_4\xi = 0$$

$$6C_4 = 1$$

$$\Rightarrow \quad C_4(\xi) = \frac{1}{6}, C_3(\xi) = -\frac{1}{2}\xi, C_2(\xi) = \frac{1}{2}\xi^2, C_1(\xi) = -\frac{1}{6}\xi^3 \tag{5}$$

From the property 4 of Green's function, it must satisfy the boundary conditions :

$$G(0, \xi) = 0, \; G'_x(0, \xi) = 0$$

$$G(1, \xi) = 0, \; G'_x(1, \xi) = 0$$

The relations reduce to

$$a_1 = 0, \; a_2 = 0$$

$$b_1 + b_2 + b_3 + b_4 = 0$$

$$b_2 + 2b_3 + 3b_4 = 0 \tag{6}$$

From the relation (4), (5) and (6), we have

$$C_1 = b_1(\xi) - a_1(\xi) \Rightarrow b_1(\xi) = -\frac{1}{6}\xi^3$$

or $$C_2 = b_2(\xi) - a_2(\xi) \Rightarrow b_2(\xi) = \frac{1}{2}\xi^2$$

or $b_3 + b_4 = \frac{1}{6}\xi^3, \frac{1}{2}\xi^2, 2b_3 + 3b_4 = -\frac{1}{2}\xi^2$

$\Rightarrow$ $b_4(\xi) = \frac{1}{2}\xi^2 - \frac{1}{3}\xi^3$ and $b_3(\xi) = \frac{1}{2}\xi^3 - \xi^2$

or $C_3(\xi) = b_3(\xi) - a_3(\xi)$

$\Rightarrow$ $a_3(\xi) = b_3(\xi) - C_3(\xi) = \frac{1}{2}\xi^3 - \xi^2 + \frac{1}{2}\xi$

and $C_4(\xi) = b_4(\xi) - a_4(\xi)$

$\Rightarrow$ $a_4(\xi) = b_4(\xi) - C_4(\xi) = \frac{1}{2}\xi^2 - \frac{1}{3}\xi^3 - \frac{1}{6}$

Substituting the value of the constants $a_1$, $a_2$, $a_3$, $a_4$, $b_1$, $b_2$, $C_3$, $C_4$ in the relation (3), the Green's function G(x, $\xi$) is obtained as

$$G(x,\xi)=\begin{cases}\left(\frac{1}{2}\xi-\xi^2+\frac{1}{2}\xi^3\right)x^2-\left(\frac{1}{6}-\frac{1}{2}\xi^2+\frac{1}{3}\xi^3\right)x^3 & , \quad 0\le x\le\xi \\ -\frac{1}{6}\xi^3+\frac{1}{2}\xi^2x+\left(\frac{1}{2}\xi^3-\xi^2\right)x^2+\left(\frac{1}{2}\xi^2-\frac{1}{3}\xi^3\right)x^3 & , \quad \xi\le x\le 1\end{cases}$$

The expression G(x, $\xi$) may be transformed to

$$G(x,\xi) = \left(\frac{1}{2}x-x^2+\frac{1}{2}x^3\right)\xi^2-\left(\frac{1}{6}-\frac{1}{2}x^2+\frac{1}{3}x^3\right)\xi^3, \quad \xi\le x\le 1$$

$\Rightarrow$ G(x, $\xi$) = G($\xi$, x), that is, Green's function is symmetric.

**3.2.4. Example.** Construct Green's function for the equation $x\frac{d^2u}{dx^2}+\frac{du}{dx} = 0$ with the conditions u(x) is bounded as $x\to 0$, $u(1) = \mu u'(1)$, $\mu \neq 0$.

**Solution.** The differential equation is given by $x\frac{d^2u}{dx^2}+\frac{du}{dx} = 0$ (1)

or $\left(\frac{d^2u/dx^2}{du/dx}\right)dx = -\frac{1}{x}dx$

or $\log\frac{du}{dx} = -\log x + \log A$

or $\frac{du}{dx} = \frac{A}{x}$

or $u(x) = A\log x + B$ (2)

The conditions u(x) is bounded as $x \to 0$ and $u(1) = \mu u'(1)$, $\mu \neq 0$ has only a trivial solution $u(x) \equiv 0$, hence we can construct a (unique) Green's function $G(x, \xi)$

Consider the function $G(x, \xi)$ as:

$$G(x, \xi) = \begin{cases} a_1 + a_2 \log x, & 0 < x \le \xi \\ b_1 + b_2 \log x, & \xi \le x \le 1 \end{cases} \quad (3)$$

where $a_1, a_2, b_1, b_2$ are unknown functions of $\xi$.

Consider $C_k = b_k(\xi) - a_k(\xi)$, k = 1, 2,…

From the continuity of $G(x, \xi)$ for $x = \xi$, we obtain

$$b_1 + b_2 \log \xi - a_1 - a_2 \log \xi = 0$$

and the jump $G'_x(x, \xi)$ at the point $x = \xi$ is equal to $\frac{1}{\xi}$ so that

$$b_2 . \frac{1}{\xi} - a_2 . \frac{1}{\xi} = -\frac{1}{\xi}$$

Putting $C_1 = b_1 - a_1$, $C_2 = b_2 - a_2$ (4)

$\Rightarrow$ $C_1 + C_2 \log \xi = 0$, $C_2 = -1$.

Hence $C_1 = \log \xi$ and $C_2 = -1$ (5)

The boundedness of the function $G(x, \xi)$ as $x \to 0$ gives $a_2 = 0$

Also, $G(x, \xi) = \mu G'_x(x, \xi)$, $b_1 = \mu b_2$

$\Rightarrow$ $a_1 = -(\mu + \log \xi)$, $a_2 = 0$, $b_1 = -1$, $b_2 = -\mu$

Substituting the value of the constants $a_1, a_2, b_1, b_2$ in the relation (3), the Green's function is obtained as

$$G(x, \xi) = \begin{cases} -(\mu + \log \xi), & 0 < x \le \xi \\ -(1 + \mu \log x), & \xi \le x \le 1 \end{cases}.$$

**3.2.5. Exercise.**

1. Construct the Green's function for the boundary value problem $u''(x) + \mu^2 u = 0$ with the conditions u(0) = u(1) = 0.

**Answer.** $G(x, \xi) = \begin{cases} \dfrac{\sin \mu(\xi - 1) \sin \mu x}{\mu \sin \mu}, & 0 \le x \le \xi \\ \dfrac{\sin \mu\xi \sin \mu(x-1)}{\mu \sin \mu}, & \xi < x \le 1 \end{cases}$

**2.** Find the Green's function for the boundary value problem $\frac{d^2u}{dx^2} - u(x) = 0$ with the conditions u(0) = u(1) = 0.

**Answer.** $G(x, \xi) = \begin{cases} \dfrac{\sinh x \sinh(\xi - 1)}{\sinh 1}, & 0 \le x \le \xi \\ \dfrac{\sinh \xi \sinh(x - 1)}{\sinh 1}, & \xi \le x \le 1 \end{cases}.$

**3.2.6. Article.** If u(x) has continuous first and second derivatives, and satisfies the boundary value problem $\frac{d^2u}{dx^2} + \lambda u = 0$ with u(0) = u(l) = 0 then u(x) is continuous and satisfies the homogeneous linear integral equation u(x) = $\lambda \int_0^1 G(x, \xi)\, u(\xi)\, d\xi$.

**Solution :** The differential equation may be written as

$$\frac{d^2u}{dx^2} + \lambda u = 0 \;\Rightarrow\; \frac{d^2u}{dx^2} = -\lambda u \qquad (1)$$

By integrating with regard to x over the interval (0, x) two times, we obtain

$$\frac{du}{dx} = -\lambda \int_0^x u(\xi)\, d\xi + C$$

or

$$u(x) = -\lambda \int_0^x (x - \xi)\, u(\xi)\, d\xi + C_x + D \qquad (2)$$

where C and D are the integration constants, to be determined by the boundary conditions.

$$u(0) = 0 \quad \Rightarrow \quad D = 0$$

$$u(l) = 0 \quad \Rightarrow \quad -\lambda \int_0^l (l - \xi)\, u(\xi)\, d\xi + Cl = 0$$

$$\Rightarrow \quad C = \frac{\lambda}{l} \int_0^l (l - \xi)\, u(\xi)\, d\xi$$

Substituting the value of the constants C and D in (2), we have

$$u(x) = -\lambda \int_0^x (x - \xi)\, u(\xi)\, d\xi + \frac{\lambda}{l} \int_0^l x(l - \xi)\, u(\xi)\, d\xi$$

or $$u(x) = -\lambda \int_0^x (x - \xi)\, u(\xi)\, d\xi + \frac{\lambda}{l} \int_0^x x(l - \xi)\, u(\xi)\, d\xi + \frac{\lambda}{l} \int_x^l x(l - \xi)\, u(\xi)\, d\xi$$

or $$u(x) = \lambda \int_0^x \frac{\xi}{l}(l - x)\, u(\xi)\, d\xi + \lambda \int_x^l \frac{x}{l}(l - \xi)\, u(\xi)\, d\xi$$

or $$u(x) = \lambda \int_0^l G(x, \xi)\, u(\xi)\, d\xi$$

where $$G(x, \xi) = \begin{cases} \frac{\xi}{l}(l-x) & , x > \xi \\ \frac{x}{l}(l-\xi) & , x < \xi \end{cases}.$$

**3.2.7. Exercise.**

1. Construct the Green's function for the boundary value problem $\frac{d^2u}{dx^2} + \mu^2 u = 0$ with the conditions u(0) = u(l) = 0.

   **Answer.** $$G(x, \xi) = \begin{cases} a_1 \cos\mu x + a_2 \sin\mu x = -\frac{\sin\mu(\xi - l)\sin\mu x}{\mu \sin\mu l} & , \quad 0 \le x < \xi \\ b_1 \cos\mu x + b_2 \sin\mu x = -\frac{\sin\mu\xi \sin\mu(x-l)}{\mu\sin\mu l} & , \quad \xi < x \le l \end{cases}$$

2. Construct the Green's function for the boundary value problem $\frac{d^2u}{dx^2}$ = 0 with the conditions u(0) = $u'(1)$ and $u'(0)$ = u(1).

   **Answer.** $$G(x, \xi) = \begin{cases} (-\xi+2)x + (-\xi+1) & , \quad 0 \le x < \xi \\ (-\xi+1)x + 1 & , \quad \xi < x \le 1 \end{cases}$$

3. Construct the Green's function for the boundary value problem $\frac{d^3u}{dx^3}$ = 0 with the boundary conditions u(0) = $u'(1)$ = 0 and $u'(0)$ = u(1).

   **Answer.** $$G(x, \xi) = \begin{cases} \frac{1}{2}x(\xi-1)[x - x\xi + 2\xi] & 0 \le x < \xi \\ \frac{1}{2}\xi[x(2-x)(\xi-2)+\xi] & \xi < x \le 1 \end{cases}$$

4. Construct the Green's function for the boundary value problem $x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} - u$ = 0 with u(x) is bounded as x$\rightarrow$0 and u(1) = 0.

   **Answer.** $$G(x, \xi) = \begin{cases} \frac{1}{2}x\left(\frac{1}{\xi^2} - 1\right) & , \quad 0 \le x < \xi \\ \frac{1}{2}\left(\frac{1}{x} - x\right) & , \quad \xi < x \le 1 \end{cases}$$

5. Construct the Green's function for the boundary value problem $\frac{d^2u}{dx^2} - u$ = 0 with the conditions u(0) = $u'(0)$ and u(l) + $\lambda$ $u'(l)$ = 0.

**Answer.** G(x, $\xi$) = $\begin{cases} -\dfrac{1}{2}\left(\dfrac{1-\lambda}{1+\lambda}\right)e^{x+\xi-2l}+\dfrac{1}{2}e^{x-\xi} \ , \ 0\le x<\xi \\ -\dfrac{1}{2}\left(\dfrac{1-\lambda}{1+\lambda}\right)e^{x+\xi-2l}+\dfrac{1}{2}e^{\xi-x} \ , \ \xi<x\le 1 \end{cases}$, where $|\lambda|\ne 1$.

6. Using Green's function, solve the boundary value problem $u''$(x) $-$u(x) = x with boundary conditions u(0) = u(1) = 0.

**Answer.** Here, G(x, $\xi$) = $\begin{cases} -\dfrac{\sinh x \sinh(\xi-1)}{\sinh 1} \ , 0\le x<\xi \\ -\dfrac{\sinh \xi \sinh(x-1)}{\sinh 1} \ , \xi < x\le 1 \end{cases}$ and the solution of the given boundary value problem is given by u(x) = $\int_0^1 G(x\,,\,\xi)\xi\, d\xi$, so u(x) = $\dfrac{\sinh x}{\sinh 1}-x$.

7. Using Green's function, solve the boundary value problem $\dfrac{d^2u}{dx^2}+u = x$ with the boundary conditions u(0) = 0 and u( $\pi/2$ ) = 0.

**Answer.** Here, G(x, $\xi$) = $\begin{cases} \cos\xi \sin x \ , \ 0\le x<\xi \\ \sin\xi\cos x \ , \ \xi<x\le\pi/2 \end{cases}$ and u(x) = $\int_0^{\pi/2} G(x\,,\,\xi)\,\xi\, d\xi$ , implies

u(x) = x $-$ $\dfrac{\pi}{2}$sin x.

8. Solve the boundary value problem using Green's function

$$\frac{d^2u}{dx^2}+u = x^2 \text{ ; } u(0) = u(\pi/2) = 0.$$

**Answer.** u(x) = $-\left[2\cos x+\sin x\left(2-\dfrac{\pi^2}{4}\right)+x^2-2\right]$.

**3.3. Construction of Green's function when the boundary value problem contains a parameter.**

Consider a differential equation of order n

$$L(u) - \lambda h = h(x) \quad (1)$$

with $V_k(u) = 0$, k = 1, 2, 3,…,n (2)

where $L(u)\equiv p_0(x)u^n(x)+p_1(x)u^{n-1}(x)+...+p_n(x)u(x)$ (3)

and $V_k(u)\equiv\alpha_k u(a)+\alpha_k^1 u'(a)+...+\alpha_k^{n-1}u^{n-1}(a)+...+\beta_k u(b)+\beta_k^1 u'(b)+...+\beta_k^{n-1}u^{n-1}(b)+...$ (4)

where the linear forms $V_1$, $V_2$,…, $V_n$ in u(a), $u'$(a), …, $u^{n-1}$(a), u(b), $u'$(b), …, $u^{n-1}$(b) are linearly independent, h(x) is a given continuous function of x, $\lambda$ is some non–zero numerical parameter.

For $h(x) \equiv 0$, the equation (1) reduces to homogeneous boundary value problem

$$L(u) = \lambda u,$$
$$V_k(u) = 0, \; k = 1, 2, 3, \ldots, n \qquad (5)$$

Those values of $\lambda$ for which the boundary value problem (5) has non trivial solutions u(x) are called the eigenvalues. The non–trivial solutions are called the associated eigen functions.

If the boundary value problem

$$L(u) = 0,$$
$$V_k(u) = 0, \; k = 1, 2, \ldots, n \qquad (6)$$

contains the Green's function G(x, $\xi$), then the boundary value problem (1) and (2) is equivalent to the Fredholm integral equation

$$u(x) = \lambda\int_a^b G(x, \xi)\, u(\xi)\, d\xi + f(x) \qquad (7)$$

where $$f(x) = \int_a^b G(x, \xi)\, h(\xi)\, d\xi \qquad (8)$$

In particular, the homogeneous boundary value problem (5) is equivalent to the homogeneous integral equation

$$u(x) = \lambda\int_a^b G(x, \xi)\, u(\xi)\, d\xi \qquad (9)$$

Since G(x, $\xi$) is a continuous kernel, therefore the Fredholm homogeneous integral equation of second kind (9) can have at most a countable number of eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ which do not have a finite limit point. For all values of $\lambda$ different from the eigen values, the non-homogeneous equation (7) has a solution for any continuous function f(x). Thus the solution is given by

$$u(x) = \lambda\int_a^b R(x, \xi; \lambda)\, f(\xi)\, d\xi + f(\xi) \qquad (10)$$

where R(x, $\xi$ ; $\lambda$) is the resolvent kernel of the kernel G(x, $\xi$). The function R(x, $\xi$ ; $\lambda$) is a meromorphic function of $\lambda$ for any fixed values of x and $\xi$ in [a, b]. The eigen values of the homogeneous integral equation (9) may by the pole of this function.

**3.3.1. Example.** Reduce the boundary value problem $\frac{d^2u}{dx^2} + \lambda u = x$, $u(0) = u(\pi/2) = 0$, to an integral equation using Green's function.

**Solution.** Consider the associated boundary value problem

$$\frac{d^2u}{dx^2} = 0 \qquad (1)$$

whose general solution is given by u(x) = Ax + B

The boundary conditions u(0) = 0, u($\pi/2$) = 0 yields only the trivial solution u(x) $\equiv$ 0. Therefore, the Green's function G(x, $\xi$) exists for the associated boundary value problem

$$G(x, \xi) = \begin{cases} a_1 x + a_2 \ , \ 0 \le x < \xi \\ b_1 x + b_2 \ , \ \xi < x \le \pi/2 \end{cases} \tag{2}$$

The Green's function G(x, $\xi$) must satisfy the following properties :

(I) The function G(x, $\xi$) is continuous at x = $\xi$, that is,

$$b_1 \xi + b_2 = a_1 \xi + a_2$$

$$\Rightarrow \quad (b_1 - a_1)\ \xi + (b_2 - a_2) = 0 \tag{3}$$

(II) The derivative G(x, $\xi$) has a discontinuity of magnitude $-\left\{\dfrac{1}{p_0(\xi)}\right\}$ at the point x = $\xi$,

that is, $$\left(\frac{\partial G}{\partial x}\right)_{x=\xi+0} - \left(\frac{\partial G}{\partial x}\right)_{x=\xi-0} = -1 \Rightarrow b_1 - a_1 = -1 \tag{4}$$

(III) The function G(x, $\xi$) must satisfy the boundary conditions

$$G(0, \xi) = 0 \qquad \Rightarrow \qquad a_2 = 0 \tag{5}$$

$$G(\pi/2, \xi) = 0 \qquad \Rightarrow \qquad b_1\left(\frac{\pi}{2}\right) + b_2 = 0 \tag{6}$$

Solving the equations (3), (4), (5) and (6), we have

$$a_1 = 1-\frac{2\xi}{\pi},\ a_2 = 0,\ b_2 = \xi,\ b_1 = -\frac{2\xi}{\pi}.$$

Substituting the value of the constants in (2), the required Green's function G(x, $\xi$) is obtained

$$G(x, \xi) = \begin{cases} \left(1-\dfrac{2\xi}{\pi}\right)x \ , \quad 0 \le x < \xi \\ \left(1-\dfrac{2x}{\pi}\right)\xi \ , \quad \xi < x \le \pi/2 \end{cases} \tag{7}$$

Consider the Green's function G(x, $\xi$) given by the relation (7) as the kernel of the integral equation, we obtain the integral equation for u(x) :

$$u(x) = f(x) - \lambda\int_0^{\pi/2} G(x, \xi)\ u(\xi)\ d\xi, \text{ where } f(x) = \int_0^{\pi/2} G(x, \xi)\ \xi\ d\xi$$

or $$f(x) = \int_0^x \left(1-\frac{2x}{\pi}\right)\xi^2\ d\xi + \int_x^{\pi/2}\left(1-\frac{2\xi}{\pi}\right)x\xi d\xi$$

or $$f(x) = \frac{1}{3}\left(1-\frac{2x}{\pi}\right)x^3 + x\left(\frac{1}{2}\xi^2 - \frac{2}{3\pi}\xi^3\right)_x^{\pi/2}$$

or $$f(x) = \frac{1}{3}x^3 - \frac{2}{3\pi}x^4 + \frac{\pi^2 x}{24} - \frac{1}{2}x^3 + \frac{2}{3\pi}x^4$$

or $$f(x) = \frac{\pi^2}{24}x - \frac{x^3}{6}$$

Thus, the given boundary value problem has been reduced to an integral equation

$$u(x) + \lambda\int_0^{\pi/2} G(x\,,\,\xi)\; u(\xi)\; d\xi \;=\; \frac{\pi^2}{24}x - \frac{1}{6}x^3.$$

**3.3.2. Exercise.**

1. Reduce the boundary value problem $\frac{d^2u}{dx^2} + xu = 1$, $u(0) = u(1) = 0$ to an integral equation.

**Answer.** $G(x,\ \xi) = \int_0^x \xi(1-x)\; d\xi + \int_x^1 x(1-\xi)\; d\xi \;=\; \frac{1}{2}x(1-x)$, and the required integral equation is $u(x) = \int_0^1 G(x\,,\,\xi)\; \xi u(\xi)\; d\xi - \frac{1}{2}x(1-x)$

2. Reduce the boundary value problem to an integral equation

$$\frac{d^2u}{dx^2} = \lambda u + 1,\ u(0) = u'(0) = 0,\ \ u''(1) = u'''(1) = 0$$

**Answer.** $u(x) = \lambda\int_0^1 G(x\,,\,\xi)\; u(\xi)\; d\xi + f(x)$, where $f(x) = \frac{1}{24}x^2(x^2 - 4x + 6)$

3. Reduce the boundary value problem $\frac{d^2u}{dx^2} + \frac{\pi^2}{4}u = \lambda u + \cos\frac{\pi x}{2}$, with $u(-1) = u(1)$ and $u'(-1) = u'(1)$ to an integral equation.

**Answer.** Here, $$G(x,\ \xi) = \begin{cases} \frac{1}{\pi}\sin\frac{\pi}{2}(x-\xi)\ , & -1 \le x < \xi \\ \frac{1}{\pi}\sin\frac{\pi}{2}(\xi - x)\ , & \xi < x \le 1 \end{cases}$$

and $$u(x) = \lambda\int_{-1}^1 G(x\,,\,\xi)\; u(\xi)\; d\xi - \left(\frac{x}{\pi}\sin\frac{\pi x}{2} + \frac{2}{\pi^2}\cos\frac{\pi x}{2}\right).$$

4. Reduce the following boundary value problems to integral equations.

(a) $u'' + \lambda u = 2x + 1$, $u(0) = u'(1)$, $u'(0) = u(1)$

(b) $u'' + \lambda u = e^x$, $u(0) = u''(0)$, $u(1) = u'(1)$.

**Answer.** (a) Here, $G(x,\ \xi) = \begin{cases} -\{(\xi-2)x+(\xi-1)\} & , \ 0 \le x < \xi \\ -\{(\xi-1)x-1\} & , \ \xi < x \le 1 \end{cases}$ and the boundary value problem reduces to the integral equation

$$u(x) = -\lambda\int_0^1 G(x\ ,\ \xi)\ u(\xi)\ d\xi - \frac{1}{6}(2x^3+3x^2-17x-5)\ .$$

(b) Here, $G(x,\ \xi) = \begin{cases} -(1+x)\xi & , \ 0 \le x < \xi \\ -(1+\xi)x & , \ \xi < x \le 1 \end{cases}$ and the boundary value problem reduces to

$$u(x) = -\lambda\int_0^1 G(x\ ,\ \xi)\ u(\xi)\ d\xi - e^x\ .$$

5. Reduce the Bessel's differential equation $x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (\lambda x^2 - 1)u = 0$ with the conditions u(0) = 0, u(1) = 0 into an integral equation.

**Answer.:** The standard equation of Bessel's equation is given by

Here, $G(x,\ \xi) = \begin{cases} \dfrac{x}{2\xi}(1-\xi^2)\ , & 0 \le x < \xi \\ \dfrac{\xi}{2x}(1-x^2)\ , & \xi < x \le 1 \end{cases}$ and the integral equation can be obtained as

$$u(x) = \lambda\int_0^1 G(x\ ,\ \xi)\ r(\xi)\ u(\xi)\ d\xi\ .$$

6. Determine the Green's function $G(x,\ \xi)$ for the differential equation $\left[\frac{d}{dx}\left(x\frac{d}{dx}\right) - \frac{n^2}{x}\right]u = 0$ with the conditions u(0) = 0 and u(1) = 0.

**Answer.** $G(x,\ \xi) = \begin{cases} \dfrac{x^n}{2n\xi^n}(1-\xi^{2n})\ , & x < \xi \\ \dfrac{\xi^n}{2nx^n}(1-x^{2n})\ , & x > \xi. \end{cases}$

**3.4. Non–homogeneous ordinary Equation.** The boundary value problem associated with a non – homogenous ordinary differential equation of second order is

$$Ly \equiv A_0(x)\frac{d^2y}{dx^2} + A_1(x)\frac{dy}{dx} + A_2(x)\,y = f(x),\ a < x < b \qquad (1)$$

with boundary conditions $\left.\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) &= 0 \\ \beta_1 y(b) + \beta_2 y'(b) &= 0 \end{aligned}\right\}$ (2)

where $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ are constants.

**3.4.1. Self–Adjoint Operator.** The operator L is said to be self – adjoint if for any two functions say u(x) and v(x) operated on L, the expression (vLu − uLv) dx is an exact differential that is, there exist a function g such that dg = (vLu − uLv) dx.

**3.4.2. Green's Function Method.** Green's function method is an important method to solve B.V.P. associated with non–homogeneous ordinary or partial differential equation . Here we shall show that a B.V.P. will be reduced to a Fredholm integral equation whose kernel is Green's function. We shall be using a special type of B.V.P. namely Sturm – Liouville's problem.

**3.4.3. Theorem.** Show that the differential operator L of the Sturm – Liouville's Boundary value problem (S.L.B.V.P.)

$$Ly = \frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + \left[q(x)+\lambda p(x)\right] y(x) = 0 \qquad (1)$$

with
$$\left.\begin{aligned}\alpha_1 y(a)+\alpha_2 y'(a) &= 0\\ \beta_1 y(b)+\beta_2 y'(b) &= 0\end{aligned}\right\} \qquad (2)$$

where $\alpha$, $\beta$, $\alpha_2$ and $\beta_2$ are constants is self adjoint.

**Proof.** Let u and v be two solutions of the given S.L.B.V.P. then

$$Lu = \frac{d}{dx}\left[r(x)\frac{du}{dx}\right] + \left[g(x)+\lambda p(x)\right] u(x) = 0$$

and
$$Lv = \frac{d}{dx}\left[r(x)\frac{dv}{dx}\right] + \left[q(x)+\lambda p(x)\right] v(x) = 0$$

So,

$$vLu - uLv = v\frac{d}{dx}\left[r(x)\frac{du}{dx}\right] + \left[q(x)+\lambda p(x)\right] u(x)v - \left[u\frac{d}{dx}\left[r(x)\frac{dv}{dx}\right]+\left[q(x)+\lambda p(x)\right]v(x)u\right]$$

$$= v\frac{d}{dx}\left[r(x)\frac{du}{dx}\right] - u\frac{d}{dx}\left[r(x)\frac{dv}{dx}\right]$$

$$= \left[v\frac{d}{dx}\left(r(x)\frac{du}{dx}\right)+\left(r(x)\frac{du}{dx}\right)\frac{dv}{dx}\right]-\left[u\frac{d}{dx}\left(r(x)\frac{dv}{dx}\right)+\left(r(x)\frac{dv}{dx}\right)\frac{du}{dx}\right]$$

$$= \frac{d}{dx}\left[r(x)\,v(x)\frac{du}{dx}\right] - \frac{d}{dx}\left[r(x)\,u(x)\frac{dv}{dx}\right]$$

$$= \frac{d}{dx}\left[r(x)\,v(x)\frac{du}{dx} - r(x)\,u(x)\frac{dv}{dx}\right] = \frac{d}{dx}\left[r(x)\left(v(x)\frac{du}{dx}-u(x)\frac{dv}{dx}\right)\right] = \frac{dg}{dx}$$

where $g = r(x)\left(v(x)\frac{du}{dx} - u(x)\frac{dv}{dx}\right)$. Then, (vLu − uLv) dx = dg

Hence operator in equation (1) is self – adjoint.

**3.4.4. Construction of Green's function by variation of parameter method.**

Consider the non – homogeneous differential equation

$$Lu = \frac{d}{dx}\left[r(x)\ \frac{du}{dx}\right] + \left[q(x) + \lambda p(x)\right]\ u(x)\ = f(x) \qquad (1)$$

subject to boundary condition:

$$\left.\begin{aligned} \alpha_1 u(a) + \alpha_2 u'(a)\ &=\ 0 \\ \beta_1 u(b) + \beta_2 u'(b)\ &=\ 0 \end{aligned}\right\} \qquad (*)$$

Construct the Green's function and show that

$$u(x) = -\int_a^b G(x\ ,\ \xi)\ \ f(\xi)\ d\xi \qquad (**)$$

where $G(x,\ \xi)$ is the Green's function defined above.

**Solution.** Let $v_1(x)$ and $v_2(x)$ be two linearly independent solution of the homogeneous differential equation.

$$Lu = \frac{d}{dx}\left[r(x)\frac{du}{dx}\right] + \ \left[q(x) + \lambda p(x)\right]\ u(x) = 0 \qquad (2)$$

Then the general solution of (2) by the method of variation of parameters is

$$u(x) = a_1(x)\ v_1(x) + a_2(x)\ v_2(x) \qquad (3)$$

where the unknown variables $a_1(x)$ and $a_2(x)$ are to be determined. We assume that neither the solution $v_1(x)$ nor $v_2(x)$ satisfy both the boundary conditions at x = a and x = b but the general solution u(x) satisfies these conditions.

Now, we differentiate (3) w.r.t. x and obtain

$$u'(x)\ =\ a_1'v_1 + a_1v_1' + a_2'v_2 + a_2v_2' \qquad (4)$$

Let us equate to zero the terms involving derivatives of parameter, that is,

$$a_1'(x)v_1(x) + a_2'(x)v_2(x)\ = 0 \qquad (5)$$

which yields

$$u'(x)\ =\ a_1(x)\ v_1'(x) + a_2(x)\ v_2'(x) \qquad (6)$$

Putting the values of u(x) and $u'(x)$ from (3) and (6) respectively in equation (1), we obtain

$$Lu = \frac{d}{dx}\left[r(x)\ \left(a_1v_1' + a_2v_2'\right)\right] + \left[q(x) + \lambda p(x)\right](a_1v_1 + a_2v_2) = f(x)$$

or $a_1\left[\frac{d}{dx}(r\,v_1') + v_1(q+\lambda p)\right] + a_2\frac{d}{dx}\left[(r\,v_2') + v_2(q+\lambda p)\right] + \left(a_1'v_1' + a_2'v_2'\right)\, r(x) = f(x)$ (7)

Since $v_1$ and $v_2$ are solutions of homogeneous equation (2), so by (7), we get

$$\left(a_1'v_1' + a_2'v_2'\right) r(x) = f(x)$$

$$\Rightarrow \quad a_1'(x)\, v_1'(x) + a_2'(x)\, v_2'(x) = \frac{f(x)}{r(x)} \tag{8}$$

Equations (5) and equation (8) can be solved to get

$$a_1'(x) = \frac{f(x)\, v_2(x)}{r(x)\left[v_2v_1' - v_1v_2'\right]} \quad \text{and} \quad a_2'(x) = \frac{-f(x)\, v_1(x)}{r(x)\left[v_2v_1' - v_1v_2'\right]} \tag{9}$$

Now the operator L is exact and we have proved that

$$v_2\, L\, v_1 - v_1\, L\, v_2 = \frac{d}{dx}\left[r(x)\,\left(v_2v_1' - v_1v_2'\right)\right] \tag{10}$$

Since $v_1$ and $v_2$ are solutions of Sturm – Liouville homogeneous differential equation so that $Lv_1 = 0$ and $Lv_2 = 0$ and thus equation (10) gives

$$\frac{d}{dx}\left[r(x)\,\left(v_2v_1' - v_1v_2'\right)\right] = 0$$

$$\Rightarrow \quad r(x)\,\left(v_2v_1' - v_1v_2'\right) = \text{constant} = -\beta \text{ (say)} \tag{11}$$

Thus, equation (9) becomes

$$a_1'(x) = \frac{-f(x)\, v_2(x)}{\beta} \quad \text{and} \quad a_2'(x) = \frac{f(x)\, v_1(x)}{\beta} \tag{12}$$

Integrating (12), we get

$$a_1(x) = -\frac{1}{\beta}\int_{c_1}^{x} f(\xi)\, v_2(\xi)\, d\xi \tag{13}$$

and $$a_2(x) = \frac{1}{\beta}\int_{c_2}^{x} f(\xi)\, v_1(\xi)\, d\xi \tag{14}$$

where $c_1$ and $c_2$ are arbitrary constants to be determined from the boundary condition on $a_1(x)$ and $a_2(x)$. These conditions are to be imposed in accordance with our earlier assumption that $v_1(x)$ and $v_2(x)$ does not satisfy boundary conditions but the final solution u(x) satisfies boundary conditions in equation (*). So, that

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0 \tag{15}$$

$$\beta_1 u(b) + \beta_2 u'(b) = 0 \tag{16}$$

Using (3) and (6) in equation (15), we obtain

$$0 = \alpha_1 u(a) + \alpha_2 u'(a)$$

$$= \alpha_1 \left[a_1(a)v_1(a) + a_2(a)v_2(a)\right] + \alpha_2 \left[a_1(a)v_1'(a) + a_2(a)v_2'(a)\right]$$

$$= a_1(a)\left[\alpha_1 v_1(a) + \alpha_2 v_1'(a)\right] + a_2(a)\left[\alpha_1 v_2(a) + \alpha_2 v_2'(a)\right]$$

Let us now assume that $v_2(x)$ satisfies first boundary condition of (*) but $v_1(x)$ does not satisfy it, then

$$\alpha_1 v_2(a) + \alpha_2 v_2'(a) = 0$$

$$\alpha_1 v_1(a) + \alpha_2 v_2'(a) \neq 0$$

so that $a_1(a)\ \left[\alpha_1 v_1(a) + \alpha_2 v_1'(a)\right] = 0 \Rightarrow a_1(a) = 0$

Using this condition in (13), we get

$$0 = a_1(a) = -\frac{1}{\beta}\int_{c_1}^{a} f(\xi)\, v_2(\xi)\ d\xi \text{ which is satisfied when } c_1 = a$$

Thus, the solution in (13) is :

$$a_1(x) = -\frac{1}{\beta}\int_{a}^{x} f(\xi)\, v_2(\xi)\ d\xi \tag{17}$$

Similarly, using (3) and (6) in (16), we obtain $c_2 = b$ and the solution in (14) is :

$$a_2(x) = \frac{1}{\beta}\int_{b}^{x} f(\xi)\, v_1(\xi)\ d\xi = -\frac{1}{\beta}\int_{x}^{b} f(\xi)\, v_1(\xi)\ d\xi \tag{18}$$

The final solution of the non – homogeneous B.V.P. is

$$u(x) = a_1(x)\, v_1(x) + a_2(x)\, v_2(x)$$

$$= -\frac{1}{\beta}\ v_1(x)\int_{a}^{x} f(\xi)\, v_2(\xi)\, d\xi - \frac{1}{\beta}\ v_2(x)\int_{x}^{b} f(\xi)\, v_1(\xi)\, d\xi$$

$$= -\int_{a}^{x} \frac{v_1(x)\, v_2(\xi)}{\beta}\ f(\xi)\, d\xi - \int_{x}^{b} \frac{v_2(x)\, v_1(\xi)}{\beta}\ f(\xi)\, d\xi = -\int_{a}^{b} G(x, \xi)\, f(\xi)\ d\xi$$

where $G(x, \xi) = \begin{cases} \dfrac{1}{\beta} v_1(x)\, v_2(\xi) & \xi \le x \le b \\ \dfrac{1}{\beta} v_2(x)\, v_1(\xi) & a \le x \le \xi \end{cases}$

### 3.5. Basic Properties of Green’s Function.

**3.5.1. Theorem.** The Green function $G(x, \xi)$ is symmetric in x and $\xi$, that is, $G(x, \xi) = G(\xi, x)$.

**Proof.** Interchanging x and $\xi$ in G(x, $\xi$) defined above :

$$G(x, \xi) = \begin{cases} \frac{1}{\beta} v_1(\xi)\, v_2(x) & x \le \xi \\ \frac{1}{\beta} v_1(x)\, v_2(\xi) & \xi \le x \end{cases} = G(\xi, x).$$

**3.5.2. Theorem.** The function G(x, $\xi$) satisfies the boundary conditions given in equation (*).

**Proof.** Consider

$$\alpha_1 G(a, \xi) + \alpha_2 G'(a, \xi) = \alpha_1 \left[\frac{v_2(a) v_1(\xi)}{\beta}\right] + \alpha_2 \left[\frac{v_2'(a) v_1(\xi)}{\beta}\right]$$

$$= \frac{1}{\beta}\left[\alpha_1 v_2(a) + \alpha_2 v_2'(a)\right] v_1(\xi)$$

$$= \frac{1}{\beta}[0] v_1(\xi) = 0 \quad x \le \xi \le b$$

Again, $\beta_1 G(b, \xi) + \beta_2 G'(b, \xi) = \beta_1 \left[\frac{v_1(b) v_2(\xi)}{\beta}\right] + \beta_2 \left[\frac{v_1'(b) v_2(\xi)}{\beta}\right]$

$$= \frac{1}{\beta}\left[\beta_1 v_1(b) + \beta_2 v_1'(b)\right] v_2(\xi)$$

$$= \frac{1}{\beta}[0] v_2(\xi) = 0 \quad , a \le \xi \le x.$$

**3.5.3. Theorem.** The function G(x, $\xi$) is continuous in [a, b]

**Proof.** Clearly, G(x, $\xi$) is continuous at every point of [a, b] except possibly at x = $\xi$. By definition of G(x, $\xi$), it can be observed that both branches have same value at x = $\xi$ given by $\frac{1}{\beta}\left[v_1(\xi)\, v_2(\xi)\right]$.

Hence G(x, $\xi$) is continuous in [a, b].

**3.5.4. Theorem.** $\frac{\partial G}{\partial x}$ has a jump discontinuity at x = $\xi$, given by

$$\left.\frac{\partial G}{\partial x}\right|_{x = \xi^+} - \left.\frac{\partial G}{\partial x}\right|_{x = \xi^-} = -\frac{1}{r(\xi)}$$

where r(x) is the co – efficient of $u''(x)$ in equation (1).

**Proof.** We have $\left.\frac{\partial G}{\partial x}\right|_{\substack{x = \xi^+ \\ (x > \xi)}} - \left.\frac{\partial G}{\partial x}\right|_{\substack{x = \xi^- \\ (x < \xi)}} = \frac{1}{\beta}\left[v_1'(x)\, v_2(\xi)\right]_{x = \xi} - \frac{1}{\beta}\left[v_2'(x)\, v_1(\xi)\right]_{x = \xi}$

$$= \frac{1}{\beta}\left[v_1'(\xi)\, v_2(\xi) - v_2'(\xi)\, v_1(\xi)\right]$$

$$= \frac{1}{\beta}\left[\frac{-\beta}{r(\xi)}\right] = -\frac{1}{r(\xi)} \qquad \text{[By equation (11)]}$$

**3.6. Fredholm Integral Equation and Green's Function.** Consider the general boundary value problem

$$A_0(x)\frac{d^2y}{dx^2} + A_1(x)\frac{dy}{dx} + A_2(x)\, y + \lambda\, p(x)\, y = h(x) \qquad (1)$$

with boundary conditions: y(a) = 0, y(b) = 0. (2)

We shall show that it reduces to Fredholm integral equation with the Green's function as its kernel.

To make the above operator in (1) as a self – adjoint operator, we shift the term $\lambda$ p(x)y to the right side and then divide it by $\frac{r(x)}{A_0(x)}$.

The solution of (1) in terms of Green's function is

$$y(x) = -\int_a^b G(x, \xi)\, f(\xi)\, d\xi \text{ where } f(x) = h(x) - \lambda\, p(x)\, y(x)$$

or
$$y(x) = -\int_a^b G(x, \xi)\, \left[h(\xi) - \lambda\, p(\xi)\, y\right] d\xi$$

$$= -\int_a^b G(x, \xi)\, h(\xi)\, d\xi + \lambda\int_a^b G(x, \xi)\, p(\xi)\, y(\xi)\, d\xi$$

$$= K(x) + \lambda \int_a^b G(x, \xi)\, p(\xi)\, y(\xi)\, d\xi \qquad (3)$$

where
$$K(x) = -\int_a^b G(x, \xi)\, h(\xi)\, d\xi \qquad (4)$$

This is a Fredholm integral equation of the second kind with kernel K(x, $\xi$) = G(x, $\xi$) p($\xi$) and a non – homogeneous term K(x).

Now, multiplying equation (3) by $\sqrt{p(x)}$, we get

$$\sqrt{p(x)}\; y(x) = \sqrt{p(x)}\; K(x) + \lambda\int_a^b \sqrt{p(x)\, p(\xi)}\; G(x, \xi)\sqrt{p(\xi)}\; y(\xi)\, d\xi$$

Let us use, $u(x) = \sqrt{p(x)}\; y(x)$ and $g(x) = \sqrt{p(x)}\; K(x)$

Then, $u(x) = g(x) + \lambda\int_a^b \sqrt{p(x)\,p(\xi)}\; G(x, \xi)\, u(\xi)\, d\xi$ (5)

Here the kernel of Fredholm integral equation of second kind is symmetric that is,

$$K(x, \xi) = \sqrt{p(x)\,p(\xi)}\; G(x, \xi) \quad (6)$$

is symmetric, since $G(x, \xi)$ is symmetric.

**Remark :** We had obtained the required result in equation (3). We had proceed to obtain equation (5) just to get the kernel in more symmetric form.

**3.7. Check Your Progress.**

1. Solve the boundary value problem using Green's function $\frac{d^2u}{dx^2} - u = -2e^x$ with boundary conditions $u(0) = u'(0)$, $u(l) + u'(l) = 0$.

**Answer.** $G(x, \xi) = \begin{cases} \frac{1}{2}e^{x-\xi} & , \; 0 \le x < \xi \\ \frac{1}{2}e^{-(x-\xi)} & , \; \xi < x \le l \end{cases}$ and $u(x) = -[(l - x)e^x + \sinh x]$

2. Solve the boundary value problem using Green's function $\frac{d^4u}{dx^4} = 1$, with boundary conditions $u(0) = u'(0) = u''(1) = u'''(1) = 0$.

**Answer.** $G(x, \xi) = \begin{cases} \frac{1}{6}x^2(3\xi - x) & , \; 0 \le x < \xi \\ \frac{1}{6}\xi^2(3\xi - x) & , \; \xi < x \le 1 \end{cases}$ and $u(x) = \frac{1}{24}x^2(x^2 - 4x + 6)$.

**3.8. Summary.** In this chapter, we discussed various methods to construct Green function for a given non-homogeneous linear second order boundary value problem and then boundary value problem can be reduced to Fredholm integral equation with Green function as kernel and hence can be solbed using the methods studied in the previous chapter.

**Books Suggested:**

1. Jerri, A.J., Introduction to Integral Equations with Applications, A Wiley-Interscience Publication, 1999.
2. Kanwal, R.P., Linear Integral Equations, Theory and Techniques, Academic Press, New York.
3. Lovitt, W.V., Linear Integral Equations, McGraw Hill, New York.
4. Hilderbrand, F.B., Methods of Applied Mathematics, Dover Publications.